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Carnot efficiencies for moving reservoirs

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Abstract. We consider a standard Carnot cycle, except that the reservoirs (of proper temperatures T_H and T_C) are in uniform motion. Two efficiencies for such cycles exist in the literature. Each has the property that it can exceed the usual Carnot efficiency $\eta_C = 1 - T_C/T_H$ for a class of inertial observers. It is shown in this paper, that a reasonable definition of efficiency leads to the usual Carnot efficiency η_C *for all inertial observers.*

1. Introduction

For many years now relativistic Carnot-type efficiencies have been available in the literature which suggest that it is in these cycles possible to do better than expected from nonrelativistic Carnot cycles. To be precise, let v_H, v_C be the constant velocities of hot and cold reservoirs respectively, both judged in some third inertial frame I. Let $\gamma_H \equiv (1 - v_H^2/c^2)^{-1/2}$, and similarly for γ_C, and let T_H, T_C be the absolute proper temperatures of the reservoirs. Then the Carnot cycle efficiency derived in the Planck–Einstein–Tolman scheme is (Tolman 1934)

$$\eta_P = 1 - \gamma_H T_C/\gamma_C T_H. \tag{1}$$

Since the nonrelativistic efficiency is

$$\eta_C = 1 - T_C/T_H \tag{2}$$

it follows that

$$\text{if } \gamma_H/\gamma_C < 1, \qquad \text{then } \eta_C < \eta_P < 1. \tag{3}$$

An inertial observer for whom the hot reservoir moves faster than the cold reservoir can therefore beat the Carnot efficiency. This strange result is independent of the controversial relativistic temperature transformation, since only proper temperatures enter in equation (1). It does depend, however, on what is considered to be a sensible definition of efficiency.

The efficiency (Moller 1967 and 1969, Ott 1963)

$$\eta_M = 1 - \gamma_C T_C/\gamma_H T_H \tag{4}$$

has also occurred, but is subject to the analogous difficulty

$$\text{if } \gamma_C/\gamma_H < 1, \qquad \text{then } \eta_C < \eta_M < 1. \tag{5}$$

This efficiency has been also discussed by Landsberg and Johns (1970), Bičák (1969) and Landsberg (1970). We shall offer arguments in favour of η_C being more meaningful in all frames of reference. This proposal of an efficiency for moving reservoirs which is the same for all inertial observers has not been made before, and the detailed arguments which lead to the results (1) and (4), as well as to (2), will therefore be presented.

2. The relativistic Carnot cycle

The relativistic Carnot cycle analysis is as follows.

(i) Start with the working fluid at rest in the rest frame I_H of the hot reservoir, so that in frame I the total energy and momentum of the fluid are in state 1

$$E_1 = \gamma_H M_0 c^2 \qquad \boldsymbol{P}_1 = \gamma_H M_0 \boldsymbol{v}_H.$$

Proper entropy and proper temperature of the fluid are S_{1H}, T_{1H}, the second suffix indicating the frame considered (the rest frame in this case). The fluid is accelerated to come to rest in the frame of the colder reservoir, the work done on the fluid being $(\gamma_C - \gamma_H)M_0c^2$. The fluid is then in state 2.

(ii) The fluid has in frame I total energy and momentum given by

$$E_2 = \gamma_C M_0 c^2 \qquad \boldsymbol{P}_2 = \gamma_C M_0 \boldsymbol{v}_C.$$

Its proper entropy is the entropy for state 1 considered in frame I_C plus the increase in entropy due to passage from state 1 to state 2, considered in frame I_C

$$S_{2C} = S_{1C} + (S_{1\to 2})_C.$$

Analogously, the entropy of the fluid in frame I_H can be written formally as

$$S_{2H} = S_{1H} + (S_{1\to 2})_H.$$

A quasistatic adiabatic expansion takes place to proper temperature T_C, the work done on the system being $\phi \Delta W_2$, say $(-1 < \phi < 0)$. The fluid is then in state 3, and an explicit expression for ϕ is not needed.

(iii) The total energy and momentum of the fluid in I are now

$$E_3 = \gamma_C(M_0c^2 + \phi\Delta W_2) \qquad \boldsymbol{P}_3 = (\boldsymbol{v}_C/c^2)E_3.$$

A quasistatic isothermal compression occurs in which heat $T_C\Delta S_C$ is gained by the system. But ΔS_C is negative so that heat is given up to the colder reservoir. The work done on the system is $\psi\Delta W_2$ (say), where the magnitude of ΔW_2 has been chosen in (ii) and (iii) so that $\phi + \psi = 1$. In frame I the heat gained by the fluid ΔQ_3, and the associated work, ΔZ_3, done on the fluid are, according to the Planck–Einstein–Tolman formation,

$$\Delta Q_3 = \frac{T_C \Delta S_C}{\gamma_C} \qquad \Delta Z_3 = \gamma_C \left(\frac{v_C}{c}\right)^2 T_C \Delta S_C$$

so that $\Delta Q_3 + \Delta Z_3$ is the total energy, $\gamma_C T_C \Delta S_C$, associated with the heat supplied, gained by the fluid as viewed in I. The total energy gained is therefore $\gamma_C(T_C\Delta S_C + \psi\Delta W_2)$ per cycle. The fluid is then in state 4.

(iv) The total energy and momentum of the fluid in I are now

$$E_4 = \gamma_C(M_0c^2 + \Delta W_2 + T_C\Delta S_C) \qquad \boldsymbol{P}_4 = (\boldsymbol{v}_C/c^2)E_4.$$

The fluid is accelerated to come to rest in I_H, the work done on the system in frame I being

$$(\gamma_H - \gamma_C)(M_0c^2 + \Delta W_2 + T_C\Delta S_C).$$

The fluid is then in state 5.

(v) The total energy and momentum of the fluid in I are

$$E_5 = \gamma_H(M_0c^2 + \Delta W_2 + T_C\Delta S_C) \qquad \boldsymbol{P}_5 = (\boldsymbol{v}_H/c^2)E_5.$$

A quasistatic adiabatic compression takes the fluid back to proper temperature T_H, the work done on the system being $\nu\Delta W_5$ (say). The fluid is then in state 6.

(vi) The total energy and momentum of the fluid in I is

$$E_6 = \gamma_H(M_0c^2 + \Delta W_2 + \nu\Delta W_5 + T_C\Delta S_C) \qquad \boldsymbol{P}_6 = (\boldsymbol{v}_H/c^2)E_6.$$

Quasistatic isothermal expansion takes place in which the system gains proper entropy ΔS_H and heat $T_H\Delta S_H$, ΔS_H being positive. The work done on the system is $\mu\Delta W_5$ (say) where ΔW_5 has been chosen in (v) and (vi) so that $\nu + \mu = 1$. In frame I the heat gained by the fluid and the associated work done on the fluid are

$$\Delta Q_6 = T_H\Delta S_H/\gamma_H \qquad \Delta Z_6 = \gamma_H(v_H/c)^2 T_H\Delta S_H$$

so that $\Delta Q_6 + \Delta Z_6$ is the total energy, $\gamma_H T_H\Delta S_H$, associated with the heat supplied, gained by the fluid as viewed in I. The energy of the fluid in I is now

$$E_7 = \gamma_H(M_0c^2 + \Delta W_2 + \Delta W_5 + T_C\Delta S_C + T_H\Delta S_H).$$

The entropy of the fluid at the end of the whole process is in frame I

$$S_1 + S_{1\to 2} + \Delta S_C + S_{4\to 5} + \Delta S_H$$

and the cyclic nature implies that

$$-(S_{1\to 2} + S_{4\to 5}) = \Delta S_C + \Delta S_H. \tag{6}$$

Assuming that acceleration does not affect the entropy, the left hand side vanishes and one has the result

$$\Delta S_C = -\Delta S_H. \tag{7}$$

The total heat intake of the fluid in one cycle in frame I is

$$Q \equiv \Delta Q_3 + \Delta Q_6 = T_C\Delta S_C/\gamma_C + T_H\Delta S_H/\gamma_H. \tag{8}$$

The work done on the fluid due to heat flow in one cycle in frame I is

$$W_Q \equiv \Delta Z_3 + \Delta Z_6 = \gamma_C T_C\Delta S_C + \gamma_H T_H\Delta S_H - Q. \tag{9}$$

The work done on the fluid due to acceleration in one cycle in frame I is

$$W_a \equiv (\gamma_H - \gamma_C)(\Delta W_2 + T_C\Delta S_C). \tag{10}$$

The work done due to compression and expansion is similarly

$$W_c \equiv \gamma_H\Delta W_5 + \gamma_C\Delta W_2. \tag{11}$$

Total work done on the fluid in frame I in one cycle is

$$W = W_Q + W_a + W_c = \gamma_H(T_H\Delta S_H + T_C\Delta S_C + \Delta W_2 + \Delta W_5) - Q. \tag{12}$$

By the first law in frame I, $W+Q = 0$, whence

$$T_H \Delta S_H + T_C \Delta S_C + \Delta W_2 + \Delta W_5 = 0 \tag{13}$$

so that $E_7 = E_1$, as required by the cyclic nature of the process.

3. Various efficiencies

The standard notion of a Carnot efficiency as work done by the fluid divided by the heat supplied by the hot reservoir has to be examined with a view to generalizing it to the present case. We present three approaches, the last of which is new, and yields a purely thermodynamic efficiency.

3.1. The Planck–Einstein approach

The work done per cycle by the fluid in I is given by (12) and the heat supplied is ΔQ_6. Hence using (12) and (13)

$$\eta_P = \frac{-W_a - W_c - W_Q}{\Delta Q_6} = \frac{Q}{\Delta Q_6} = 1 + \frac{\Delta Q_3}{\Delta Q_6} = 1 + \frac{\gamma_H}{\gamma_C}\frac{T_C}{T_H}\frac{\Delta S_C}{\Delta S_H}. \tag{14}$$

Use of (7) yields (1).

3.2. The Ott–Møller approach

First observe that using (10) to (13)

$$W_a + W_c = (\gamma_H - \gamma_C) T_C \Delta S_C + \gamma_H (\Delta W_2 + \Delta W_5) = -\gamma_H T_H \Delta S_H - \gamma_C T_C \Delta S_C. \tag{15}$$

One now denies the existence of the work terms W_Q due to heat flow so that ΔQ_6 in (14) is replaced by the total energy given up by the hot reservoir. Hence using (15)

$$\eta_M = \frac{-W_a - W_c}{\Delta Q_6 + \Delta Z_6} = 1 + \frac{\gamma_C T_C \Delta S_C}{\gamma_H T_H \Delta S_H}. \tag{16}$$

Use of (7) yields (4).

3.3. The present approach

If one imagines a mechanical engine as being coupled to the Carnot engine, it can perform mechanical work $W_a + W_c$ on the fluid in frame I in each cycle; put differently it can absorb work $-(W_a + W_c)$ from the fluid. However, the work term W_Q is entirely generated by heat transfer between fluid and reservoir without involving the engine. This work is hence not utilizable, and ought therefore not to appear in the numerator of the efficiency. The origin of a work term which does not involve the engine is due to the change of rest mass of a fluid by heat transfer. This implies changes of momentum in frame I, and hence corresponds to work having been done, though no engine was involved.

Part of the work $-(W_a + W_c)$ has, however, not come from the thermal energy of the hot reservoir, but from the kinetic energy of both reservoirs. This contribution may be estimated most simply by imagining an initial process, prior to the Carnot cycle, in which both reservoirs are accelerated to their final speeds from a state of rest in I.

One also requires a final process in which the reservoirs are decelerated to rest in frame I. The work done on the reservoirs in these two processes is

$$W_R \equiv (\gamma_H - 1)M_{H0}c^2 + (\gamma_C - 1)M_{C0}c^2 - \{(\gamma_H - 1)(M_{H0}c^2 - T_H\Delta S_H) + (\gamma_C - 1)(M_{C0} - T_C\Delta S_C)\}$$
$$= (\gamma_H - 1)T_H\Delta S_H + (\gamma_C - 1)T_C\Delta S_C.$$

The reservoirs have therefore contributed mechanical work $-W_R$, so that the whole system has performed mechanical work $-(W_a + W_c + W_R)$. The initial and final state of the whole system are now similar mechanical states in I, namely states of zero kinetic energy. The only difference is a thermodynamic one, namely the changes in heat content of the reservoirs. The source of potentially utilizable energy being the hot reservoir, the denominator of the efficiency must be $T_H\Delta S_H$. We therefore arrive at

$$\eta = \frac{-W_a - W_c - W_R}{T_H\Delta S_H} = 1 + \frac{T_C\Delta S_C}{T_H\Delta S_H}.$$

Use of (7) yields the normal Carnot efficiency for any frame I.

If one does not wish to add the processes of initial acceleration and final deceleration of the reservoirs to rest in I, the above result can still be used. The argument takes then the following slightly different form: the work done by the fluid is $-(W_a + W_c)$ but this amount incorporates the purely mechanical work W_R effectively done on the reservoirs. This must therefore be subtracted, leaving one again with $-(W_a + W_c + W_R)$ as the work term due to the thermodynamic processes, for use in the numerator of the efficiency.

The fact that the efficiencies η_P, η_M previously calculated depend on the frame of reference I can therefore be attributed to the fact that the mechanical work effectively done on the reservoirs during the cycle depends on this frame. It is in agreement with physical intuition that reservoir motion, and certainly observer motion, should leave the purely thermodynamic efficiency unaltered, as it does in our approach (§ 3.3) where allowance has been made for all mechanical effects.

Note that $\eta = 0$ is a necessary condition for thermal equilibrium if any one of the three efficiencies is used. It is not a sufficient condition because thermal equilibrium requires $T_C = T_H$ and $v_C = v_H$. It appears on the contrary that in the general relativistic Carnot cycle (Ebert and Göbel, to be published in *General Relativity and Gravitation*) $\eta = 0$ is both necessary and sufficient for thermal equilibrium.

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